

# RULED CR-SUBMANIFOLDS OF LOCALLY CONFORMAL KÄHLER MANIFOLDS

GABRIEL EDUARD VÎLCU

ABSTRACT. The purpose of this paper is to study the canonical totally real foliations of CR-submanifolds in a locally conformal Kähler manifold.

2010 *Mathematics Subject Classification*: 53C15.

*Keywords*: locally conformal Kähler structure, ruled submanifold, CR-submanifold, distribution, foliation.

## 1. INTRODUCTION

The concept of CR-submanifold, first introduced in Kähler geometry by A. Bejancu [3], was later considered and studied in locally conformal Kähler ambient by many authors (see e.g. [1, 7, 8, 10, 11, 14, 18, 19, 20, 21, 23]). Such a submanifold comes naturally equipped with some canonical foliations, which were first investigated by B.Y. Chen and P. Piccinni [9] (see also Chapter 12 from the monograph [13]). One of these foliations, denoted by  $\mathfrak{F}^\perp$  and called the totally real foliation, is given by the totally real distribution involved in the definition of the CR-submanifold, proven to be always completely integrable by D.E. Blair and B.Y. Chen [5]. On the other hand, A. Bejancu and H.R. Farran [4, Chapter 5] investigated the relationship between the geometry of the totally real foliation on a CR-submanifold of a Kähler manifold and the geometry of the CR-submanifold itself, stressing on the links between the foliation and the complex structure on the embedding manifold (see also the monograph [2] for an excellent survey concerning foliations in CR geometry). Moreover, they also used the theory of ruled submanifolds (see [22] for a detailed survey on the topic) to characterize some classes of CR-submanifolds in Kähler manifolds. At the end of the chapter, the authors have proposed, as an interesting and useful research, the extension of this study to CR-submanifolds of manifolds endowed with various geometric structures. This was done recently for quaternionic and paraquaternionic Kähler ambient [15, 16, 25]. In this paper, following the same techniques, we study the CR-submanifolds in a locally conformal Kähler manifold. In particular, we obtain necessary and sufficient conditions for a CR-submanifold of a locally conformal Kähler manifold to be ruled with respect to the totally real foliation  $\mathfrak{F}^\perp$ . In the last part of the paper characterizations are provided for this foliation to become Riemannian, i.e. with bundle-like metric.

## 2. PRELIMINARIES

Let  $(\overline{M}, J, \overline{g})$  be an almost Hermitian manifold of dimension  $2n$ , where  $J$  denotes the almost complex structure and  $\overline{g}$  the Hermitian metric. Then  $(\overline{M}, J, \overline{g})$  is called a *locally conformal Kähler* (briefly l.c.K.) manifold if for each point  $p$  of  $\overline{M}$  there

exists an open neighbourhood  $U$  of  $p$  and a positive function  $f_U$  on  $U$  so that the local metric

$$\bar{g}_U = \exp(-f_U)\bar{g}|_U$$

is Kählerian (see [17, 24]). If  $U = \bar{M}$ , then the manifold  $(\bar{M}, J, \bar{g})$  is said to be a *globally conformal Kähler* (briefly g.c.K.) manifold. Equivalently (see [13]),  $(\bar{M}, J, \bar{g})$  is l.c.K. if and only if there exists a closed 1-form  $\omega$ , globally defined on  $\bar{M}$ , such that

$$d\Omega = \omega \wedge \Omega,$$

where  $\Omega$  is the Kähler 2-form associated with  $(J, \bar{g})$ , i.e.

$$\Omega(X, Y) = \bar{g}(X, JY),$$

for  $X, Y \in \Gamma(T\bar{M})$ . The 1-form  $\omega$  is called the *Lee form* and its metrically equivalent vector field  $B = \omega^\sharp$ , where  $\sharp$  means the rising of the indices with respect to  $\bar{g}$ , namely

$$\bar{g}(X, B) = \omega(X),$$

for all  $X \in \Gamma(T\bar{M})$ , is called *Lee vector field*. It is known that  $(\bar{M}, J, \bar{g})$  is globally conformal Kähler (respectively Kähler) if the Lee-form  $\omega$  is exact (respectively  $\omega = 0$ ). It is also known that Levi-Civita connections  $D^U$  of the local metrics  $\bar{g}_U$  glue up to a globally defined torsion free linear connection  $D$  on  $\bar{M}$ , called the *Weyl connection* of the l.c.K. manifold  $\bar{M}$ , given by

$$D_X Y = \bar{\nabla}_X Y - \frac{1}{2} [\omega(X)Y + \omega(Y)X - \bar{g}(X, Y)B]$$

for any  $X, Y \in \Gamma(T\bar{M})$ , where  $\bar{\nabla}$  is the Levi-Civita connection of  $\bar{g}$ . Moreover, Weyl connection  $D$  satisfies  $D\bar{g} = \omega \otimes \bar{g}$  and  $DJ = 0$ . As a consequence, considering the anti-Lee form  $\theta = \omega \circ J$  and the anti-Lee vector field  $A = -JB$ , one can obtain a third equivalent definition in terms of the Levi-Civita connection  $\bar{\nabla}$  of the metric  $\bar{g}$  (see [13]). Namely,  $(\bar{M}, J, \bar{g})$  is l.c.K. if and only if the following equation is satisfied for any  $X, Y \in \Gamma(T\bar{M})$ :

$$(\bar{\nabla}_X J)Y = \frac{1}{2} [\theta(Y)X - \omega(Y)JX - \bar{g}(X, Y)A - \Omega(X, Y)B]. \quad (1)$$

A submanifold  $M$  of a l.c.K. manifold  $(\bar{M}, J, \bar{g})$  is called a *CR-submanifold* if there exists a differentiable distribution  $D : p \rightarrow D_p \subset T_p M$  on  $M$  satisfying the following conditions:

- i.  $D$  is holomorphic, i.e.  $JD_p = D_p$ , for each  $p \in M$ ;
- ii. the complementary orthogonal distribution  $D^\perp : p \rightarrow D_p^\perp \subset T_p M$  is totally real, i.e.  $JD_p^\perp \subset T_p^\perp M$  for each  $p \in M$ .

If  $\dim D_p^\perp = 0$  (resp.  $\dim D_p = 0$ ), then the CR-submanifold is said to be a *holomorphic* (resp. a *totally real*) submanifold. A CR-submanifold is called a *proper* CR-submanifold if it is neither holomorphic nor totally real.

*Remark 2.1.* Let  $M$  be a CR-submanifold of a l.c.K. manifold  $(\bar{M}, J, \bar{g})$ . By the definition of a CR-submanifold we have the orthogonal decomposition

$$TM = D \oplus D^\perp.$$

Also, the normal bundle has the orthogonal decomposition

$$TM^\perp = JD^\perp \oplus \mu,$$

where  $\mu$  is the subbundle of the normal bundle  $TM^\perp$  which is the orthogonal complement of  $JD^\perp$ . Corresponding to the last decomposition, any normal vector field  $N$  can be written as  $N = N_{JD^\perp} + N_\mu$ , where  $N_{JD^\perp}$  (resp.  $N_\mu$ ) is the  $JD^\perp$ – (resp.  $\mu$ –) component of  $N$ . It is easy to see that the subbundle  $\mu$  is invariant under the action of  $J$ . We note that if  $\mu = 0$ , then the CR-submanifold is said to be an *anti-holomorphic* submanifold or a *generic* submanifold.

If we denote by  $\nabla$  the Levi-Civita connection on  $(M, g)$ , where  $g$  is the induced Riemannian metric by  $\bar{g}$  on  $M$ , then the Gauss and Weingarten formulas are given by:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM) \quad (2)$$

and

$$\bar{\nabla}_X N = -a_N X + \nabla_X^\perp N, \quad \forall X \in \Gamma(TM), \quad \forall N \in \Gamma(TM^\perp) \quad (3)$$

where  $h$  is the second fundamental form of  $M$ ,  $\nabla^\perp$  is the connection on the normal bundle and  $a_N$  is the shape operator of  $M$  with respect to  $N$ . It is well-known that  $h$  is a symmetric  $F(M)$ –bilinear form and  $a_N$  is a self-adjoint operator, related by:

$$g(a_N X, Y) = \bar{g}(h(X, Y), N) \quad (4)$$

for all  $X, Y \in \Gamma(TM)$  and  $N \in \Gamma(TM^\perp)$ . We say (see [4]) that the distribution  $D$  (resp.  $D^\perp$ ) is  $a_N$ –invariant, if  $a_N X \in \Gamma(D)$  (resp.  $a_N Z \in \Gamma(D^\perp)$ ) for any  $X \in \Gamma(D)$  (resp.  $Z \in \Gamma(D^\perp)$ ).

A CR-submanifold  $M$  of a l.c.K. manifold  $(\bar{M}, J, \bar{g})$  is called:

- i. *D-geodesic* if  $h(X, Y) = 0, \forall X, Y \in \Gamma(D)$ .
- ii.  *$D^\perp$ -geodesic* if  $h(X, Y) = 0, \forall X, Y \in \Gamma(D^\perp)$ .
- iii. *mixed geodesic* if  $h(X, Y) = 0, \forall X \in \Gamma(D), Y \in \Gamma(D^\perp)$ .

We recall now the following result which we shall need in the sequel.

**Theorem 2.2.** *Let  $M$  be a CR-submanifold of a l.c.K. manifold  $(\bar{M}, J, \bar{g})$ . Then:*

- i. *The totally real distribution  $D^\perp$  is integrable [5].*
- ii. *The holomorphic distribution  $D$  is integrable if and only if*

$$\bar{g}(h(X, JY), JZ) = \bar{g}(h(JX, Y), JZ) - \Omega(X, Y)\theta(Z)$$

*for all  $X, Y \in \Gamma(D)$  and  $Z \in \Gamma(D^\perp)$  [6].*

### 3. TOTALLY REAL FOLIATION OF A CR-SUBMANIFOLD IN A LOCALLY CONFORMAL KÄHLER MANIFOLD

Let  $M$  be a CR-submanifold of a l.c.K. manifold  $(\bar{M}, J, \bar{g})$ . From Theorem 2.2 we have that the distribution  $D^\perp$  is always integrable and gives rise to a foliation of  $M$  by totally-real submanifolds of  $\bar{M}$ . So any CR-submanifold of a l.c.K. manifold comes naturally equipped with a foliation denoted by  $\mathfrak{F}^\perp$  and called the *totally real foliation*. We note that if the holomorphic distribution  $D$  is also integrable, then  $M$  carries a foliation by holomorphic submanifolds of  $\bar{M}$ , called the *Levi foliation* (see [6, 12]).

We recall that if each leaf of a foliation  $\mathfrak{F}$  on  $M$  is a totally geodesic submanifold of  $M$ , then we say that  $\mathfrak{F}$  is a *totally geodesic foliation*. Next we state some characterizations of totally geodesic totally real foliations on CR-submanifolds.

**Proposition 3.1.** *The canonical totally real foliation  $\mathfrak{F}^\perp$  on a CR-submanifold  $M$  of a l.c.K. manifold  $(\overline{M}, J, \overline{g})$  is a totally geodesic foliation if and only if*

$$\theta(Y)JX = 2h_{JD^\perp}(X, Y), \quad \forall X \in \Gamma(D^\perp), Y \in \Gamma(D). \quad (5)$$

*Proof.* For  $X, Z \in \Gamma(D^\perp)$  and  $Y \in \Gamma(D)$ , using (1)–(4), we derive:

$$\begin{aligned} \overline{g}(J\nabla_X Z, Y) &= -\overline{g}(\nabla_X Z, JY) \\ &= -\overline{g}(\overline{\nabla}_X Z - h(X, Z), JY) \\ &= \overline{g}(-(\overline{\nabla}_X J)Z + \overline{\nabla}_X JZ, Y) \\ &= -\frac{1}{2}\overline{g}(\theta(Z)X - \omega(Z)JX - g(X, Z)A - \Omega(X, Y)B - 2\overline{\nabla}_X JZ, Y) \\ &= -\frac{1}{2}\overline{g}(g(X, Z)JB - 2\overline{\nabla}_X JZ, Y) \\ &= \frac{1}{2}g(X, Z)\overline{g}(B, JY) + \overline{g}(-a_{JZ}X + \nabla_X^\perp JZ, Y) \\ &= \frac{1}{2}g(X, Z)\omega(JY) - \overline{g}(h(X, Y), JZ) \\ &= \frac{1}{2}\overline{g}(\theta(Y)JX, JZ) - \overline{g}(h(X, Y), JZ). \end{aligned}$$

Therefore, we obtain

$$\overline{g}(J\nabla_X Z, Y) = \frac{1}{2}\overline{g}(\theta(Y)JX - 2h_{JD^\perp}(X, Y), JZ), \quad \forall X, Z \in \Gamma(D^\perp), Y \in \Gamma(D). \quad (6)$$

If we suppose now  $\mathfrak{F}^\perp$  is a totally geodesic foliation, then  $\nabla_X Z \in \Gamma(D^\perp)$ , for all  $X, Z \in \Gamma(D^\perp)$ , and from (6) we deduce:

$$\overline{g}(\theta(Y)JX - 2h_{JD^\perp}(X, Y), JZ) = 0, \quad \forall Z \in \Gamma(D^\perp)$$

and the implication follows.

Conversely, if we suppose  $\theta(Y)JX = 2h_{JD^\perp}(X, Y)$ , for all  $X \in \Gamma(D^\perp)$ ,  $Y \in \Gamma(D)$ , then from (6) we derive:

$$\overline{g}(J\nabla_X Z, Y) = 0$$

and we conclude  $\nabla_X Z \in \Gamma(D^\perp)$ . Thus  $\mathfrak{F}^\perp$  is a totally geodesic foliation.  $\square$

*Remark 3.2.* An alternative proof of the above Proposition can be obtained using [6, Lemma 1, p. 343].

**Theorem 3.3.** *Let  $M$  be a CR-submanifold of a l.c.K. manifold  $(\overline{M}, J, \overline{g})$  such that the Lee vector field  $B$  is normal to  $M$ . Then the next assertions are equivalent:*

- i. *The canonical totally real foliation  $\mathfrak{F}^\perp$  on  $M$  is totally geodesic.*
- ii.  *$h(X, Y) \in \Gamma(\mu)$ ,  $\forall X \in \Gamma(D^\perp)$ ,  $Y \in \Gamma(D)$ .*
- iii. *The totally real distribution  $D^\perp$  is  $a_N$ -invariant for any  $N \in \Gamma(JD^\perp)$ .*
- iv. *The holomorphic distribution  $D$  is  $a_N$ -invariant for any  $N \in \Gamma(JD^\perp)$ .*

*Proof.* Since  $B$  is normal to  $M$ , we deduce

$$\theta(Y) = \omega(JY) = \overline{g}(JY, B) = 0$$

for any  $Y \in \Gamma(D)$ . Therefore, from the above Proposition we obtain (i)  $\Leftrightarrow$  (ii).

The equivalence of (ii) and (iii) follows easily from (4), while the equivalence of (iii) and (iv) holds because  $a_N$  is a self-adjoint operator.  $\square$

*Remark 3.4.* We note that Theorem 3.3 extends Theorem 4.1 in [4, p. 247] from the case of an ambient Kählerian manifold to the case of an ambient l.c.K. manifold.

**Corollary 3.5.** *Let  $M$  be a CR-submanifold of a l.c.K. manifold  $(\overline{M}, J, \overline{g})$  such that the Lee vector field  $B$  is normal to  $M$ . Then:*

- i. *If  $M$  is mixed geodesic, then the totally real foliation  $\mathfrak{F}^\perp$  on  $M$  is totally geodesic.*
- ii. *If  $M$  is an anti-holomorphic submanifold, then  $M$  is mixed geodesic if and only if the totally real foliation  $\mathfrak{F}^\perp$  is totally geodesic.*

*Proof.* The proof is clear from Theorem 3.3.  $\square$

*Remark 3.6.* We note that the Corollary 3.5(i.) has been also obtained using a different proof by Dragomir [10] (see also [13, Theorem 12.6, p. 168]). On the other hand, Corollary 3.5(ii.) gives us an interesting geometric characterization of mixed geodesic anti-holomorphic submanifolds in a l.c.K. manifold normal to the Lee vector field. Thus,  $M$  is mixed geodesic if and only if any geodesic of a leaf of  $D^\perp$  is a geodesic of  $M$ . On another hand, according to Corollary 3.5(i.), if  $M$  is totally geodesic, then  $M$  is mixed geodesic and any geodesic of a leaf of  $\mathfrak{F}^\perp$  is a geodesic of  $M$  which in turn is a geodesic of  $\overline{M}$ . Therefore any leaf of  $\mathfrak{F}^\perp$  is totally geodesic immersed in  $(\overline{M}, J, \overline{g})$ . It is important to note that this property is also true in Kähler ambient (see [4, Corollary 4.4, p. 148]).

A submanifold  $M$  of a Riemannian manifold  $(\overline{M}, \overline{g})$  is said to be a *ruled submanifold* if it admits a foliation whose leaves are totally geodesic immersed in  $(\overline{M}, \overline{g})$ . A CR-submanifold which is a ruled submanifold with respect to the canonical foliation  $\mathfrak{F}^\perp$  is called a *totally real ruled CR-submanifold*. We are able now to state the following characterization of totally real ruled CR-submanifolds in l.c.K. manifolds.

**Theorem 3.7.** *Let  $M$  be a CR-submanifold of a l.c.K. manifold  $(\overline{M}, J, \overline{g})$ . Then the next assertions are equivalent:*

- i.  *$M$  is a totally real ruled CR-submanifold.*
- ii.  *$M$  is  $D^\perp$ -geodesic and the anti-Lee form  $\theta$  and the second fundamental form  $h$  of the submanifold are related by (5).*
- iii. *The second fundamental form  $h$ , the anti-Lee form  $\theta$  and the anti-Lee vector field  $A$  are related by (5) and satisfy:*

$$h(X, Z) \in \Gamma(\mu), \quad \forall X, Z \in \Gamma(D^\perp) \quad (7)$$

and

$$(\nabla_X^\perp JZ)_\mu = -\frac{1}{2}g(X, Z)A_\mu, \quad \forall X, Z \in \Gamma(D^\perp), \quad (8)$$

where the index  $\mu$  denotes the  $\mu$ -component of the vector field.

*Proof.* i.  $\Leftrightarrow$  ii. For any  $X, Z \in \Gamma(D^\perp)$  we have:

$$\begin{aligned} \overline{\nabla}_X Z &= \nabla_X Z + h(X, Z) \\ &= \nabla_X^{D^\perp} Z + h^{D^\perp}(X, Z) + h(X, Z) \end{aligned}$$

and thus we conclude that the leaves of  $D^\perp$  are totally geodesic immersed in  $\overline{M}$  if and only if  $h^{D^\perp} = 0$  and  $M$  is  $D^\perp$ -geodesic. The equivalence follows now easily from Proposition 3.1.

i.  $\Leftrightarrow$  iii. For  $X, Z \in \Gamma(D^\perp)$ , and  $U \in \Gamma(D)$  we obtain similarly as in the proof of Proposition 3.1:

$$\begin{aligned}\bar{g}(\bar{\nabla}_X Z, U) &= \bar{g}(J\bar{\nabla}_X Z, JU) \\ &= \bar{g}(-(\bar{\nabla}_X J)Z + \bar{\nabla}_X JZ, JU) \\ &= \frac{1}{2}\bar{g}(\theta(JU)JX - 2h_{JD^\perp}(X, JU), JZ).\end{aligned}\quad (9)$$

On the other hand, if  $X, Z, W \in \Gamma(D^\perp)$ , then taking account of (2) we deduce:

$$\begin{aligned}\bar{g}(\bar{\nabla}_X Z, JW) &= \bar{g}(\nabla_X Z + h(X, Z), JW) \\ &= \bar{g}(h(X, Z), JW).\end{aligned}\quad (10)$$

If we consider now  $X, Z \in \Gamma(D^\perp)$  and  $N \in \Gamma(\mu)$ , then making use of (1) and (3) we derive:

$$\begin{aligned}\bar{g}(\bar{\nabla}_X Z, N) &= \bar{g}(J\bar{\nabla}_X Z, JN) \\ &= \bar{g}(-(\bar{\nabla}_X J)Z + \bar{\nabla}_X JZ, JN) \\ &= -\frac{1}{2}\bar{g}((\theta(Z)X - \omega(Z)JX - g(X, Z)A - \Omega(X, Z)B) - 2\bar{\nabla}_X JZ, JN) \\ &= \frac{1}{2}\bar{g}(g(X, Z)A + 2\bar{\nabla}_X JZ, JN) \\ &= \frac{1}{2}\bar{g}(g(X, Z)A + 2\nabla_X^\perp JZ, JN)\end{aligned}$$

and thus we obtain:

$$\bar{g}(\bar{\nabla}_X Z, N) = \frac{1}{2}\bar{g}(g(X, Z)A_\mu + 2(\nabla_X^\perp JZ)_\mu, JN). \quad (11)$$

Finally,  $M$  is a totally real ruled CR-submanifold of  $(\bar{M}, J, \bar{g})$  if and only if  $\bar{\nabla}_X Z \in \Gamma(D^\perp)$ ,  $\forall X, Z \in \Gamma(D^\perp)$  and by using (9), (10) and (11) we deduce the equivalence.  $\square$

**Corollary 3.8.** *If  $M$  is a CR-submanifold of a l.c.K. manifold  $(\bar{M}, J, \bar{g})$  such that  $B \in \Gamma(JD^\perp)$ , then the next assertions are equivalent:*

- i.  $M$  is a totally real ruled CR-submanifold.
- ii.  $M$  is  $D^\perp$ -geodesic and the second fundamental form satisfies

$$h(X, Y) \in \Gamma(\mu), \quad \forall X \in \Gamma(D^\perp), \quad Y \in \Gamma(D).$$

- iii. The subbundle  $JD^\perp$  is  $D^\perp$ -parallel, i.e:

$$\nabla_X^\perp JZ \in \Gamma(JD^\perp), \quad \forall X, Z \in \Gamma(D^\perp)$$

and the second fundamental form satisfies

$$h(X, Y) \in \Gamma(\mu), \quad \forall X \in \Gamma(D^\perp), \quad Y \in \Gamma(TM).$$

- iv. The shape operator satisfies

$$a_{JZ}X = 0, \quad \forall X, Z \in \Gamma(D^\perp)$$

and

$$a_N X \in \Gamma(D), \quad \forall X \in \Gamma(D^\perp), \quad N \in \Gamma(\mu).$$

*Proof.* The equivalence of (i.), (ii.) and (iii.) is clear from the above theorem, since for any  $Y \in \Gamma(D)$  we have

$$\theta(Y) = g(JY, B) = 0.$$

The equivalence of (ii) and (iii.) follows from (4).  $\square$

**Corollary 3.9.** *Let  $M$  be a CR-submanifold of a l.c.K. manifold  $(\overline{M}, J, \overline{g})$  such that Lee vector field  $B$  is normal to  $M$ . If  $M$  is totally geodesic, then  $M$  is a totally real ruled CR-submanifold.*

*Proof.* The assertion is clear from Theorem 3.7.  $\square$

#### 4. FOLIATIONS WITH BUNDLE-LIKE METRIC ON CR-SUBMANIFOLDS OF LOCALLY CONFORMAL KÄHLER MANIFOLDS

Let  $(M, g)$  be a Riemannian manifold and  $\mathfrak{F}$  a foliation on  $M$ . The metric  $g$  is said to be bundle-like for the foliation  $\mathfrak{F}$  if the induced metric on the transversal distribution  $\mathcal{D}^\perp$  is parallel with respect to the intrinsic connection on  $\mathcal{D}^\perp$ . This is true if and only if the Levi-Civita connection  $\nabla$  of  $(M, g)$  satisfies (see [4]):

$$g(\nabla_{Q^\perp Y} QX, Q^\perp Z) + g(\nabla_{Q^\perp Z} QX, Q^\perp Y) = 0, \quad \forall X, Y, Z \in \Gamma(TM), \quad (12)$$

where  $Q^\perp$  (resp.  $Q$ ) is the projection morphism of  $TM$  on  $\mathcal{D}^\perp$  (resp  $D$ ).

If for a given foliation  $\mathfrak{F}$  there exists a Riemannian metric  $g$  on  $M$  which is bundle-like for  $\mathfrak{F}$ , then we say that  $\mathfrak{F}$  is a Riemannian foliation on  $(M, g)$ .

In what follows we provide necessary and sufficient conditions for the induced metric on a CR-submanifold of a l.c.K. manifold to be bundle-like for the totally real foliation  $\mathfrak{F}^\perp$ .

**Theorem 4.1.** *If  $M$  is a CR-submanifold of a l.c.K. manifold  $(\overline{M}, J, \overline{g})$ , then the next assertions are equivalent:*

- i. *The induced metric  $g$  on  $M$  is bundle-like for the canonical totally real foliation  $\mathfrak{F}^\perp$ .*
- ii. *The second fundamental form  $h$  of the submanifold and anti-Lee vector field  $A$  satisfy:*

$$g(U, V)A + h(U, JV) + h(V, JU) \in \Gamma(TM) \oplus \Gamma(\mu),$$

for any  $U, V \in \Gamma(D)$ .

*Proof.* From (12) we deduce that  $g$  is bundle-like for the canonical totally real foliation  $\mathfrak{F}^\perp$  if and only if:

$$g(\nabla_U X, V) + g(\nabla_V X, U) = 0, \quad \forall X \in \Gamma(D^\perp), \quad U, V \in \Gamma(D). \quad (13)$$

On the other hand, using (1)-(4), we obtain for any  $X \in \Gamma(D^\perp)$ ,  $U, V \in \Gamma(D)$ :

$$\begin{aligned}
g(\nabla_U X, V) + g(\nabla_V X, U) &= \bar{g}(\bar{\nabla}_U X - h(U, X), V) + \bar{g}(\bar{\nabla}_V X - h(V, X), U) \\
&= \bar{g}(\bar{\nabla}_U X, V) + \bar{g}(\bar{\nabla}_V X, U) \\
&= \bar{g}(-(\bar{\nabla}_U J)X + \bar{\nabla}_U JX, JV) + \bar{g}(-(\bar{\nabla}_V J)X + \bar{\nabla}_V JX, JU) \\
&= -\frac{1}{2}\bar{g}(\theta(X)U - \omega(X)JU - g(U, X)A - \Omega(U, X)B - 2\bar{\nabla}_U JX, JV) \\
&\quad -\frac{1}{2}\bar{g}(\theta(X)V - \omega(X)JV - g(V, X)A - \Omega(V, X)B - 2\bar{\nabla}_V JX, JU) \\
&= -\frac{1}{2}\bar{g}(\theta(X)U - \omega(X)JU - 2\bar{\nabla}_U JX, JV) \\
&\quad -\frac{1}{2}\bar{g}(\theta(X)V - \omega(X)JV - 2\bar{\nabla}_V JX, JU) \\
&= \omega(X)g(U, V) + \bar{g}(\bar{\nabla}_U JX, JV) + \bar{g}(\bar{\nabla}_V JX, JU) \\
&= \omega(X)g(U, V) - g(A_{JX}U, JV) - g(A_{JX}V, JU) \\
&= \omega(X)g(U, V) - \bar{g}(h(U, JV), JX) - \bar{g}(h(V, JU), JX).
\end{aligned}$$

and taking into account that  $B = \omega^\sharp$  and  $A = -JB$  we derive:

$$g(\nabla_U X, V) + g(\nabla_V X, U) = -\bar{g}(g(U, V)A + h(U, JV) + h(V, JU), JX). \quad (14)$$

The proof is now complete from (13) and (14).  $\square$

**Corollary 4.2.** *Let  $M$  be a CR-submanifold of a l.c.K. manifold  $(\bar{M}, J, \bar{g})$ .*

- i. *If  $B \in \Gamma(D) \oplus \Gamma(TM^\perp)$ , then the induced metric  $g$  on  $M$  is bundle-like for the canonical totally real foliation  $\mathfrak{F}^\perp$  if and only if*

$$h(U, JV) + h(V, JU) \in \Gamma(\mu), \quad \forall U, V \in \Gamma(D).$$

- ii. *If  $B$  has a non-vanishing component in  $\Gamma(D^\perp)$ , then the induced metric  $g$  on  $M$  is not bundle-like for the canonical totally real foliation  $\mathfrak{F}^\perp$ .*

*Proof.* The proof follows from Theorem 4.1.  $\square$

**Corollary 4.3.** *If  $M$  is an anti-holomorphic submanifold of a l.c.K. manifold  $(\bar{M}, J, \bar{g})$ , normal to the Lee field of  $\bar{M}$ , then the induced metric  $g$  on  $M$  is bundle-like for the canonical totally real foliation  $\mathfrak{F}^\perp$  if and only if*

$$h(U, JV) + h(V, JU) = 0, \quad \forall U, V \in \Gamma(D).$$

*Proof.* The assertion is clear from the above Corollary.  $\square$

#### ACKNOWLEDGEMENT

This work was supported by CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0118.

#### REFERENCES

- [1] E. Barletta, *CR submanifolds of maximal CR dimension in a complex Hopf manifold*, Ann. Global Anal. Geom. 22 (2002), No. 2, 99-118.
- [2] E. Barletta, S. Dragomir, K.L. Duggal, *Foliations in Cauchy-Riemann geometry*, Mathematical Surveys and Monographs, Vol. 140, American Mathematical Society, 2007.
- [3] A. Bejancu, *CR submanifolds of a Kaehler manifold*. I, Proc. Am. Math. Soc. 69 (1978), 135-142.



- [4] A. Bejancu, H.R. Farran, *Foliations and geometric structures*, Mathematics and Its Applications, Springer, 2006.
- [5] D.E. Blair, B.Y. Chen, *On CR submanifolds of Hermitian manifolds*, Israel J. Math. 34 (1979), 353-369.
- [6] D.E. Blair, S. Dragomir, *CR products in locally conformal Kähler manifolds*, Kyushu J. Math. 56 (2002), No. 2, 337-362.
- [7] V. Bonanzinga, K. Matsumoto, *Warped product CR-submanifolds in locally conformal Kaehler manifolds*, Period. Math. Hungar. 48 (2004), No. 1-2, 207-221.
- [8] J.L. Cabrerizo, M. Fernandez Andres, *CR-submanifolds of a locally conformal Kähler manifold*, Differential geometry (Santiago de Compostela, 1984), Res. Notes in Math. 131 (1985), Pitman, Boston, MA, 17-32.
- [9] B.Y. Chen, P. Piccinni, *The canonical foliations of a locally conformal Kähler manifold*, Ann. Mat. Pura Appl. (4) 141 (1985), 289-305.
- [10] S. Dragomir, *Cauchy-Riemann submanifolds of locally conformal Kaehler manifolds*. I-II, Geom. Dedicata 28 (1988), No. 2, 181-197; Sem. Mat. Fis. Univ. Modena XXXVII (1989), 1-11.
- [11] S. Dragomir, R. Grimaldi, *Cauchy-Riemann submanifolds of locally conformal Kaehler manifolds*. III, Serdica 17 (1991), No. 1, 3-14.
- [12] S. Dragomir, S. Nishikawa, *Foliated CR manifolds*, J. Math. Soc. Japan 56 (2004), No. 4, 1031-1068.
- [13] S. Dragomir, L. Ornea, *Locally conformal Kähler geometry*, Progress in Math. 155, Birkhäuser, Boston, Basel, 1998.
- [14] K.L. Duggal, R. Sharma, *Totally umbilical CR-submanifolds of locally conformal Kaehler manifolds*, Math. Chronicle 16 (1987), 79-83.
- [15] S. Ianuș, A.M. Ionescu, G.E. Vilcu, *Foliations on quaternion CR-submanifolds*, Houston J. Math. 34 (2008), No. 3, 739-751.
- [16] S. Ianuș, S. Marchiafava, G.E. Vilcu, *Paraquaternionic CR-submanifolds of paraquaternionic Kähler manifolds and semi-Riemannian submersions*, Cent. Eur. J. Math. 8 (2010), No. 4, 735-753.
- [17] P. Libermann, *Sur les structures presque complexes et autres structures infinitésimales régulières*, Bull. Soc. Math. France 83 (1955), 195-224.
- [18] K. Matsumoto, *On CR-submanifolds of locally conformal Kaehler manifold*. I-II, J. Korean Math. Soc. 21(1) (1984), 49-61; Tensor, N.S., 45 (1987), 144-150.
- [19] M. Munteanu, *Doubly warped product CR-submanifolds in locally conformal Kähler manifolds*, Monatsh. Math. 150 (2007), No. 4, 333-342.
- [20] L. Ornea, *On CR submanifolds of locally conformal Kaehler manifolds*, Demonstratio Math. 19 (1986), No. 4, 863-869.
- [21] N. Papaghiuc, *Some remarks on CR-submanifolds of a locally conformal Kaehler manifold with parallel Lee form*, Publ. Math. Debrecen 43 (1993), No. 3-4, 337-341.
- [22] V. Rovenski, *Foliations on Riemannian manifolds and submanifolds*, Birkhäuser Boston, Inc., Boston, MA, 1998.
- [23] B. Şahin, R. Güneş, *CR-submanifolds of a locally conformal Kaehler manifold and almost contact structure*, Math. J. Toyama Univ. 25 (2002), 13-23.
- [24] I. Vaisman, *On locally conformal almost Kähler manifolds*, Israel J. Math. 24 (1976), 338-351.
- [25] G.E. Vilcu, *Riemannian foliations on quaternion CR-submanifolds of an almost quaternion Kähler product manifold*, Proc. Indian Acad. Sci., Math. Sci. 119 (2009), No. 5, 611-618.

Gabriel Eduard VÎLCU<sup>1,2</sup>

<sup>1</sup>University of Bucharest,  
Research Center in Geometry, Topology and Algebra,  
Str. Academiei, Nr. 14, Sector 1,  
București 70109-ROMANIA  
e-mail: gvilcu@gta.math.unibuc.ro

<sup>2</sup>Petroleum-Gas University of Ploiești,  
Department of Mathematical Economics,  
Bd. București, Nr. 39,  
Ploiești 100680-ROMANIA  
e-mail: gvilcu@upg-ploiesti.ro